

ON COBWEB ADMISSIBLE SEQUENCES

The Production Theorem

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Summary

In this note further clue decisive observations on cobweb admissible sequences are shared with the audience. In particular an announced proof of the Theorem 1 (by Dziemiańczuk) from [1] announced in India -Kolkata- December 2007 is delivered here. Namely here and there we claim that any cobweb admissible sequence F is at the point product of primary cobweb admissible sequences taking values one and/or certain power of an appropriate primary number p .

Here also an algorithm to produce the family of all cobweb-admissible sequences i.e. the Problem 1 from [1] i.e. one of several problems posed in source papers [2, 3] is solved using the idea and methods implicitly present already in [4].

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http://ii.uwb.edu.pl/akk/sem/sem_rota.htm

1 Preliminaries

The notation from [2, 3, 1] is being here taken for granted.

Definition 1 ([2, 3, 1]) *A sequence F is called cobweb-admissible iff for any $n, k \in \mathbb{N} \cup \{0\}$*

$$\binom{n}{k}_F = \frac{n_F \cdot (n-1)_F \cdot \dots \cdot (n-k+1)_F}{1_F \cdot 2_F \cdot \dots \cdot k_F} \in \mathbb{N} \quad (1)$$

Problem 1 ([2, 3, 1]) *Find effective characterizations and/or an algorithm to produce the cobweb admissible sequences i.e. find all examples.*

2 Primary cobweb admissible sequence

Throughout this paper we shall consequently use p letter only for primary numbers.

Definition 2 A cobweb admissible sequence $P(p) \equiv \{n_P\}_{n \geq 0}$ valued one and/or powers of one certain primary number p i.e. $n_P \in \{1, p, p^2, p^3, \dots\}$ is called primary cobweb admissible sequence.

Theorem 1 Any cobweb-admissible sequence F is at the point product of primary cobweb-admissible sequences $P(p)$.

PROOF

Given any cobweb admissible sequence $F = \{n_F\}_{n \geq 0}$, each of its elements can be represented as a product of primary numbers' powers i.e. $n_F = \prod_{s \geq 1} p_s^{\alpha(n,s)}$. Therefore the sequence F is at the point product of sequences $P(p_1), P(p_2), \dots$ such that $P(p_s) \equiv \{n_{P_s}\}_{n \geq 0}$ and $n_{P_s} = p_s^{\alpha(n,s)}$. Each of primary sequences $P(p_s)$ where $s = 1, 2, 3, \dots$ is cobweb admissible as following holds for any $n, k \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} \binom{n}{k}_{P(p_1)} \cdot \binom{n}{k}_{P(p_2)} \cdot \dots &= \binom{n}{k}_F \in \mathbb{N} \Rightarrow \\ \Rightarrow \binom{n}{k}_{P(p_s)} &= \frac{n_{P(p_s)}^k}{k_{P(p_s)}!} = \frac{p_s^N}{p_s^K} \in \mathbb{N} \end{aligned} \quad (2)$$

where N stands for the sum of index powers' of primary numbers p_s in $n_F, (n-1)_F, \dots, (n-k+1)_F$ product expansion via primary numbers and correspondingly K is the index powers' sum for k first elements of the sequence F ■

3 Primary cobweb admissible sequences family

In this section we define a family $\mathcal{A}(p)$ of all primary cobweb admissible sequences taking values one and/or certain power of an appropriate primary number p . In the next part of this section we present the family in the graph structure of a tree defined in algorithmic way in what follows.

For this aim let consider a primary cobweb admissible sequence $F \equiv P(p)$ and its corresponding family of sequences $B(F) \equiv \{n_{B(F)}\}_{n \geq 0}$ such that

$$n_{B(F)} = m \leftrightarrow n_F = p^m$$

In the sequel we shall consider sequences for arbitrary but fixed one primary number p , therefore we use abbreviatio $P(p) \equiv P$.

Lemma 1 *Natural number valued sequence $F \equiv \{n_F\}_{n \geq 0}$ is primary cobweb admissible $P(p)$ iff for any natural number n , $n_F \in \{1, p, p^2, p^3, \dots\}$ and*

$$\forall_{1 \leq k \leq \lfloor n/2 \rfloor} \sum_{s=n-k+1}^n s_{B(F)} \geq \sum_{s=1}^k s_{B(F)}.$$

PROOF

The first steep. Given any primary cobweb admissible sequence $F \equiv \{n_F\}_{n \geq 0}$. From the Definition 2 we know that $n_F \in \{1, p, p^2, \dots\}$ for certain primary number p . From the Definition 1 we readily infer that for any $n, k \in \mathbb{N} \cup \{0\}$

$$\binom{n}{k}_F = \frac{p^{n_{B(F)}} \cdot \dots \cdot p^{(n-k+1)_{B(F)}}}{p^{1_{B(F)}} \cdot p^{2_{B(F)}} \cdot \dots \cdot p^{k_{B(F)}}} = \frac{p^N}{p^K} \in \mathbb{N} \Rightarrow N \geq K$$

where $N = \sum_{s=n-k+1}^n s_{B(F)}$ and $K = \sum_{s=1}^k s_{B(F)}$.

The second steep. Given any sequence $F \equiv \{n_F\}_{n \geq 0}$ where $n_F \in \{1, p, p^2, p^3, \dots\}$ and $\forall_{1 \leq k \leq \lfloor n/2 \rfloor} \sum_{s=n-k+1}^n s_{B(F)} \geq \sum_{s=1}^k s_{B(F)}$ (*). Then for any natural n, k below takes place

$$\binom{n}{k}_F = \frac{p^{n_{B(F)}} \cdot \dots \cdot p^{(n-k+1)_{B(F)}}}{p^{1_{B(F)}} \cdot p^{2_{B(F)}} \cdot \dots \cdot p^{k_{B(F)}}} = C \wedge (*) \Rightarrow C \in \mathbb{N}$$

■

Definition 3 (Primary cobweb admissible tree) *Let $G(p)$ to be a weighted tree $G(p) = \langle V, E, \delta \rangle$ where V stays for set of vertices, E denotes a set of nodes and function δ which assigns weight for any vertex $v \in V$ such that $\delta(v) \in \{0, 1, 2, \dots\}$ and p - primary number. We shall define the corresponding graph $G(p)$ via the following recurrence:*

1. $v_0 \in V$ is called the root with weight $\delta(v_0) = 0$
2. If $(v_0, v_1, \dots, v_{n-1})$ is a path of graph $G(p)$ then $(v_0, v_1, \dots, v_{n-1}, v_n)$ is too if, and only if $\forall_{1 \leq k \leq \lfloor n/2 \rfloor} N_{n,k} \geq K_k$

where $N_{n,k} = \sum_{i=n-k+1}^n \delta(v_i)$ and $K_k = \sum_{i=1}^k \delta(v_i)$.

Conclusion 1

Any path (v_0, v_1, \dots, v_n) from the root v_0 to vertex v_n encodes the first n terms with 0_F of primary cobweb admissible sequence $F \equiv \{n_F\}_{n \geq 0}$, $n_F \in \{1, p, p^2, \dots\}$ with help of elements' exponent powers' sequence $B(P)$ such that $k_{B(F)} = \delta(v_k)$ i.e. the $n+1$ -tuple

$$(v_0, v_1, \dots, v_n) \in V^{n+1} \leftrightarrow (\delta(v_0), \delta(v_1), \dots, \delta(v_n))$$

exactly encodes finite primary cobweb admissible sequence F valued by one and/or powers of primary number p .

Observation 1 *If any path (v_0, v_1, \dots, v_n) encodes n the first terms with 0_F of primary cobweb admissible sequence F then there exists infinite number of successors vertices v_{n+1} which encode primary cobweb admissible sequence F' specified by these n first terms with 0_F and the one additional $(n+1)_{F'} = \delta(v_{n+1})$ term.*

PROOF

If any path (v_0, v_1, \dots, v_n) encodes n the first terms with 0_F of primary cobweb admissible sequence F then there exists infinite number of natural numbers M such that $\delta(v_{n+1}) = M$ and $N_{n,k-1} + M \geq K_k$.

Consequently, now we present an algorithm to generate primary cobweb admissible tree.

Algorithm 1 (primary cobweb-admissible tree) *We shall begin with the root v_0 of graph $G(p)$ from Definition 3 and in the next steps, from any path (v_0, v_1, \dots, v_n) we obtain the very next one $(v_0, v_1, \dots, v_n, v_{n+1})$.*

Input: Any path (v_0, v_1, \dots, v_n) of $G(p)$ which encodes n the first terms with 0_F of primary cobweb admissible sequence F .

Output: Non-empty set $\emptyset \neq \Delta_n \subseteq \{v_{n+1} : \delta(v_{n+1}) \in \{0, 1, 2, \dots\}\}$ with vertices' successors for v_n vertex such that the paths $(v_0, v_1, \dots, v_n, v_{n+1})$ where $v_{n+1} \in \Delta_{n+1}$ encodes primary cobweb-admissible sequence, too.

Under the convenient notation for vertices $v(s) \equiv v_{n+1} \wedge \delta(v_{n+1}) = s$ note now the following.

Steps:

1. If $n = 1$
 $\Delta_1 = \{v(0), v(1), v(2), \dots\}$
2. If $n = 2$
 $\Delta_2 = \{v(m), v(m+1), v(m+2), \dots\}$, where $m = \delta(v_1)$
- ...
- n. For any natural n
 $\Delta_n = \{v(m), v(m+1), v(m+2), \dots\}$,
where $m = \max\{K_k - N_{n-1,k-1} : k = 1, 2, 3, \dots, \lfloor n/2 \rfloor\}$

where $K_k = \sum_{i=1}^k \delta(v_i)$ and $N_{n,k} = \sum_{i=n-k+1}^n \delta(v_i)$.

Definition 4 *Denote with letter $\mathcal{A}(p)$ the family of all primary cobweb admissible sequences $P(p)$.*

Observation 2 *The family $\mathcal{A}(p)$ is labelled-designated by the set of infinite paths (v_0, v_1, v_2, \dots) of graph $G(p)$ from the root v_0 i.e.*

$$F \in \mathcal{A}(p) \Leftrightarrow (v_0, v_1, v_2, \dots) \text{ is a path of graph } G(p)$$

where $F \equiv \{n_F\}_{n \geq 0}$ and $n_F = p^{\delta(v_n)}$.

PROOF

This is a conclusion on graph $G(p)$ (Definition 3).

The first steep. If given any primary cobweb admissible sequence $F \equiv \{n_F\}_{n \geq 0}$, $n \in \{1, p, p^2, \dots\}$, then from the Definition 1 of admissibility and the Definition 3 of tree $G(p)$ for any natural numbers n, k the following is true

$$\binom{n}{k}_F = \frac{p^{n_{B(F)}} \cdot \dots \cdot p^{(n-k+1)_{B(F)}}}{p^{1_{B(F)}} \cdot p^{2_{B(F)}} \cdot \dots \cdot p^{k_{B(F)}}} = \frac{p^N}{p^K} = \frac{p^{\delta(v_n)} \cdot \dots \cdot p^{\delta(v_{n-k+1})}}{p^{\delta(v_1)} \cdot \dots \cdot p^{\delta(v_k)}} \in \mathbb{N}$$

where $s_{B(F)} = \delta(v_s)$ from Conclusion 1. In view of the Definition 1 $N \geq K$ hence (v_0, v_1, v_2, \dots) is a path of graph $G(p)$.

The second steep. Take any given path (v_0, v_1, v_2, \dots) of the graph $G(p)$. Then by definition for any natural number n, k , $N_{n,k} \geq K_k$ where $N_{n,k} = \sum_{i=n-k+1}^n \delta(v_i)$ and $K_k = \sum_{i=1}^k \delta(v_i)$. Hence this path does encode the very primary cobweb admissible sequence $P(p)$ ■

Theorem 2 (Cobweb Admissible Sequences Production Theorem)
The family of all cobweb admissible sequences is a product of families $\mathcal{A}(p_s)$ for $s = 1, 2, 3, \dots$ i.e. for any cobweb admissible sequence F

$$F \in \times_{s=1} \mathcal{A}(p_s)$$

PROOF

This is the summarizing conclusion. Any cobweb admissible sequence F is at the point product of primary cobweb admissible sequences $P(p)$ (Theorem 1) and the family of all primary cobweb admissible sequences $\mathcal{A}(p)$ is defined by primary cobweb admissible tree $G(p)$ (Observation 2) ■

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